Noncommutative Topological Quantum Field Theory-Noncommutative Floer Homology

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Abstract

We present some ideas for a possible Noncommutative Floer Homology. The geometric motivation comes from an attempt to build a theory which applies to practically every 3-manifold (closed, oriented and connected) and not only to homology 3-spheres. There is also a physical motivation: one would like to construct a noncommutative topological quantum field theory. The two motivations are closely related since in the commutative case at least, Floer Homology Groups are part of a certain (3+1)-dim Topological Quantum Field Theory.

Classification: theoretical physics, mathematical physics, geometric topology, differential geometry, quantum algebra

Keywords: Floer Homology, Noncommutative Geometry, Topological Quantum Field Theory, 3-manifolds, K-Theory, Quantum Gravity

1 Introduction and Motivation

This article describes some ideas which emerged during out visit at the IHES in 2002. Since then progress has been slow so we decided to put these notes

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on the web hoping that they may attract some attention and someone might sed some light on certain interesting we believe issues raised hereby. It is more like a research project than a complete article.

Our motivation is twofold: it comes both from geometry and from physics. Let us start with the first, geometry. We would like to deepen our understanding on 3-manifolds. Floer Homology is a very useful device since it is the only known homology theory which is only *homeomorphism* and *not homotopy* invariant. Yet computations are particularly hard and the theory itself is very complicated; moreover the notorius reducible connections make things even worse and at the end Floer Homology Groups are defined only for homology 3-spheres. We would like to have a hopefully simpler theory which would apply to a larger class of 3-manifolds. We shall elaborate more on this in the next sections.

Our second motivation comes from physics, to be more specific we are interested in *quantum gravity* and *unifying theories*. To begin with, most physicists take the point of view that quantum gravity-which is currently an elusive theory-*should exist;* the argument in favour of its existence goes as follows (the original argument was due to P.A.M. Dirac): let us consider Einstein's classical field equations which describe gravity (we assume no cosmological constant and we set the speed of light $c = 1$):

$$G_{\mu\nu} = 8\pi G T_{\mu\nu}$$

In the above equation, G denotes Newton's constant, $T_{\mu\nu}$ denotes the energy-momentum tensor and $G_{\mu\nu}$ denotes the Einstein tensor which is equal, by definition, to $G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu}$, where $g_{\mu\nu}$ is the Riemannian metric, $R_{\mu\nu}$ is the Ricci curvature tensor and R is the scalar curvature. One can see clearly that the RHS of the above equation,namely the energy-momentum tensor, contains mass and energy coming from the other two interactions in nature; mass for instance, consists of fermions (quarks) and leptons and we know that these interactions (strong and electroweak) are quantized and hence the RHS of the equation contains *quantized quantities.* So for *consistency* of the equations, the LHS, which encodes geometry, *should also be quantized.*

[Here we would like to mention a small coment as an Aside: one may argue that the LHS may remain classical while the RHS may involve the

average value of an operator; however such a theory will not be essentially different from classical general relativity and probably not qualified to be called quantum gravity, what we have in mind is Ehrenfert Theorem from Quantum Mechanics. We think of the above field equations as describing, in the quantum level, an actual equality between operators].

Now the famous and very well-known *Holography Principle*, which has attracted a lot of attention since 1992 when it was proposed originally by G. 't Hooft (see [4]), states that quantum gravity should be a *topological quantum field theory* as defined by Atiyah in [1]. One could expect that to happen, even without holography because of the following simple fact: given (for simplicity) a closed Riemannian 4-manifold and the Einstein-Hilbert action which contains the square root of the scalar curvature of the metric as Lagrangian density, in order to compute the partition function of the theory one would have to integrate over all metrics. It is clear that if one was able to perform this functional integral, the result should be a topological invariant of the underlying manifold simply because "there is nothing else left" apart from the topology of the Riemannian manifold. We take for brevity the 4-manifold to be closed, so Atiyah's axioms for a topological quantum field theory will reduce to obtaining numerical invariants and not elements of a vector space associated to the boundary (eg Floer Homology Groups of the boundary 3-manifold). But here there is an impotant question: the partition function of the Einstein-Hilbert action on a Riemannian manifold should be a topological invariant, but should it be a *diffeomorphism or a homeomorhism* (or even homotopy) invariant? We know from the stunning work of S.K. Donaldson in the '80s (see [8]) that the DIFF and the TOP categories *in dimension 4* are two entirely different worlds (existence of "exotic" $\mathbf{R}^4$'s)! So particularly for the case of 4-manifolds (which is our intuitive idea for spacetime) this question is crucial. [We would like to make a remark here: in physics literature the term "topological" really means "metric independent" without further specification but for 4-dim geometry, this point is particularly important]. We do not have a definite answer on this but it is an issue which in most cases it is not addressed to in the literature; however we feel that for quantum physics TOP should be more appropriate as the working category since for example in quantum mechanics (solutions of Schrodinger equation) one requires only continuity and not smoothness of solutions at points connecting different regions.

But this is not enough; if we want a unifying theory of all interactions, we must have other fields present apart from the metric (eg gauge fields for electroweak and strong interactions and/or matter fields). We know from the case of the *Quantum Hall Effect* and Bellissard's work (and others') (see eg [2]) that the existence of *external fields* "make things *noncommutative*". For the particular case of the QHE the presence of a uniform magnetic field turns the Brilluin zone of a periodic crystal from a 2-torus to a noncommutative 2-torus. Further evidence for this phenomenon, namely the appearence of noncommutative spaces when external fields are present, comes from string theory: the Connes-Douglas-Schwarcz article ([5]) indicates that when a constant 3-form C (acting as a potential) of D=11 supergravity is turned on, M-theory admits additional compactifications on noncommutative tori. Also in string theory, the Seiberg-Witten article (see [6]) also discusses noncommutative effects on open strings arising from a nonzero B-field. So we believe there is good motivation to try to see what a possible *noncommutative topological quantum field theory* should look like since from what we mentioned above, it is reasonable to expect that a unifying quantum theory should have some noncommutativity arising from the extra gauge or other fields present; it should also be a topological quantum field theory since it should contain quantum gravity.

2 Topological Invariants for 3-manifolds

Let us start by recalling some well-known facts from 3-manifold topology: we fix a nice Lie group G, say $G = SU(2)$; if M is a 3-manifold with fundamental group $\pi_1(M)$, then the set

$$R(M) := Hom(\pi_1(M), G)/ad(G)$$

consisting of equivalence classes of representations of the fundamental group $\pi_1(M)$ of M onto the Lie group G modulo conjugation *tends to be discrete*. If M is a homology 3-sphere, ie $H_1(M; \mathbf{Z}) = 0$, (this is a safficient condition but not in any way necessary), then $R(M)$ has a *finite* number of elements and the *trivial representation* is *isolated*.

There is a well-known 1:1 correspondence between the elements of the set $R(M)$ and elements of the set

$A(M) := \{$*flat* G-connections on $M\}/$(gauge equivalence)

The bijection is nothing other than the *holonomy* of the flat connections.

Although $R(M)$ *depends on the homotopy type of* M, we can get *topological invariants* of M, ie invariants under *homoeomorphisms*, if we use the moduli space $A(M)$: depending on how we "decorate" the elements of $A(M)$, namely by giving different "labels" to the elements of $A(M)$, we can get the following *topological invariants* for the 3-manifold M:

1. *The (semi-classical limit of the) Jones-Witten invariant.*
Pick $G = O(n)$ and for each (gauge equivalence class of) flat $O(n)$-connection a say on M, we have a flat $O(n)$-bundle E over M with flat $O(n)$-connection a along with its exterior covariant derivative denoted d_a; now since a is flat, $d_a^2 = 0$ and hence we can form the *twisted de Rham complex* of M by the flat connection a denoted $(\Omega^*(M,E), d_a)$, where $\Omega^*(M,E)$ denotes smooth E-valued differential forms on M. We equip M with a Riemanian metric and we also define the *twisted Laplace operator* by the flat connection a to be: $\Delta_a := d_a^* d_a + d_a d_a^*$. Then the *Ray-Singer analytic torsion* $T(M,a)$ is a *non-negative real number* defined by the formula (see [9]):

$$log[T(M,a)] := \frac{1}{2}\sum_{i=0}^{3}(-1)^i i \zeta'_{\Delta_{i,a}}(0)$$

where $\Delta_{i,a}$ denotes the twisted Laplace operator acting on i-forms and

$$\zeta'_{\Delta_{i,a}}(0) := -\frac{d}{ds}\zeta_{\Delta_{i,a}}|_{s=0} = logD(\Delta_{i,a})$$

and where we call $D(\Delta_{i,a})$ the ζ-function regularised determinant of the Laplace operator $\Delta_{i,a}$ (this is a generalisation of the logarithm of the determinant of a self-adjoint operator).
The ζ-function of the Laplace operator ζ_{Δ_i} is by definition (for $s \in \mathbf{C}$):

$$\zeta_{\Delta_i}(s) := \sum_{\{\lambda_n \geq 0\}} \lambda_n^{-s} = \frac{1}{\Gamma(s)} \int_0^\infty t^{s-1} Tr(e^{-t\Delta_i}) dt$$

for $Re(s)$ large. Then ζ_{Δ_i} extends to a meromorphic function of s which is analytic at $s = 0$.

The Ray-Singer analytic torsion is independent of the Riemannian metric if the twisted de Rham cohomology groups are trivial.

If M is a homology 3-sphere (or any other 3-manifold such that the set $A(M)$ has finite cardinality), then if we sum-up the Ray-Singer analytic torsions of all the flat connections (since these are finite in number we know the sum will converge), what we shall get as a result is a topological invariant of the 3-manifold which is closely related to the "low energy limit" (or the semi-classical limit) of the *Jones-Witten* (or Reshetikin-Turaev) quantum invariants for 3-manifolds (see [7]). More precisely the low energy limit of the Jones-Witten quantum invariants for homology 3-spheres is a finite sum of combinations of the Ray-Singer torsions with the corresponding Chern-Simons numbers (ie the integral of the Chern-Simons 3-form over the compact 3-manifold M) of the flat connections.

2. *The Casson invariant.*
Let M be a homology 3-sphere and pick $G = SU(2)$. If we choose a Hegaard splitting on M, then assuming that $R(M)$ is regular (ie that the 1st twisted de Rham cohomology groups vanish for all flat connections), then each element of $R(M)$ aquires an orientation, namely a "label" +1 or -1. Let us denote by c_- (resp c_+) the number of elements of $R(M)$ with orientation -1 (resp +1). Both c_- and c_+ depend on the Hegaard splitting chosen but their *difference* $c := c_- - c_+$ *does not* (in fact it behaves like an index) and this integer c is the *Casson invariant* of the 3-manifold M. Clearly c is well defined since the cardinality of $R(M)$ is *finite* and hence both c_- and c_+ are finite.

3. *Floer Homology Groups.*
Again M is a homology 3-sphere (and hence both $R(M)$ and $A(M)$ have a finite number of elements); we pick $G = SU(2)$, we denote by $B(M)$ the

space of *all* $SU(2)$-connections on M modulo gauge transformations and we denote by $B^*(M)$ the *irreducible* ones (a connection is irreducible if its stabiliser equals the centre of $SU(2)$ where the stabiliser is the centraliser of the holonomy group of a connection). We want to do *Morse Theory* on the ∞-dim Banach manifold $B(M)$:
(i). We find a suitable "Morse function" $I : B^*(M) \to \mathbf{R}$: this is the integral over M of the Chern-Simons 3-form

$$I(A) = \frac{1}{8\pi^2} \int_M Tr(A \wedge dA + \frac{2}{3} A \wedge A \wedge A)$$

with a *finite* number of *critical points*; these are precisely the elements of $A(M)$. This is true since the solutions of the Euler-Lagrange equations for the Chern-Simons action are the flat connection 1-forms.
(ii). Then each element of $A(M)$ aquires a "label" which is the *Morse index* of the critical point; in ordinary finite dim Morse theory this is equal to the number of negative eigenvalues of the Hessian. But the Hessian of the Chern-Simons function is unbounded below and we get ∞ as Morse index for every critical point. So naive immitation of ordinary finite dim Morse theory techniques do not work.

Floer in [3] observed the following crucial fact: pick a Riemannian metric on M and considering the noncompact 4-manifold $\mathbf{R} \times M$ along with its corresponding Riemannian metric, a continuous 1-parameter family of connections A_t on M corresponds to a unique connection $\mathbf{A}$ on $\mathbf{R} \times M$; then, choosing the axial gauge (0th component of the connection vanishes), the *gradient flow* equation for the Chern-Simons function I on M corresponds to the *instanton equation* on the noncompact 4-manifold $\mathbf{R} \times M$:

$$\partial_t A_t = *F_{A_t} \Leftrightarrow F^+_{\mathbf{A}} = 0.$$

Then consider the *linearised* instanton equation $d_{\mathbf{A}} a = 0$, where a is a small perturbation. This operator is not elliptic; we perturb it to $D_{\mathbf{A}} = -d^*_{\mathbf{A}} \oplus d^+_{\mathbf{A}}$ to make it elliptic. Then the *finite* integer Morse index for each critical point comes as the relative (with respect to the trivial flat connection) Fredholm index of the perturbed elliptic operator $D_{\mathbf{A}}$. In this way the moduli space $A(M)$ aquires a $\mathbf{Z}/8$ grading and then we follow ideas from Morse theory:
(iii). We define the Floer-Morse complex using as generators the critical

points and the "differential" is essentially defined by the flow lines of the critical points. Taking the cohomology in the usual way we get the *Floer homology groups* of M. The Euler characteristic of the Floer-Morse complex equals twice the Casson invariant (see [8]).

Remarks:

(a). The structure Lie group $SU(2)$ can be replaced by another group, say $U(2)$.

(b). We assumed that all critical points were not only *non-degenerate* (i.e. $H^1_A(M) = 0$, this denotes the first twisted de Rham cohomology group of M by the flat connection A), but in fact *acyclic* (i.e. $H^0_A(M) = H^1_A(M) = 0$). If this is not the case, then the theory just becomes more complicated and one has to use *weighted spaces*.

(c). One needs a restriction of the form $b^+ > 1$ in order to be able to prove independence on the choice of the Riemannian metric (the Riemannian metric defines a Hodge star operator whose square equals 1, hence its eigenvalues are ± 1; this gives a splittting of the space of 2-forms into positive and negative eigenspaces and b^+ simply denotes the *positive* part of the 2nd Betti number).

(d). Reducible connections create more severe problems; this is the main reason why people usually work with homology 3-spheres: apart from having a finite number of gauge equivalence classes of flat connections, they have a unique reducible connection which is the trivial flat connection which is moreover isolated. If one wants to take the reducible connections into account as well, then one has to use *equivariant* Floer homology. This is a lot more complicated and less satisfactory as a theory since equivariant Floer Homology groups may be *infinite dimensional* and hence there is no Euler characteristic for the equivariant Morse-Floer complex; also there is no Casson invariant known in this case.

All the above depend crucially on the fact that $R(M)$ (or equivalently $A(M)$) has finite cardinality; the most convenient case that this is guaranteed is if M is a homology 3-sphere. So the question is: what happens if M is such that $R(M)$ does not have finite cardinality? Is there a chance to define the analogue of the Casson invariant say in this case or even more than that, a Floer homology?

We believe *"yes"* and this is precisely the point we are trying to develope here.

The key idea is the following: we want to replace $R(M)$ by another more stable and better behaving moduli spcace. To do that we use as our basis a recent result by David Gabai (see [12]): For practically any 3-manifold M (closed, oriented and connected), the moduli space $N(M)$ of taut codim-1 foliations modulo coarse isotopy has *finite* cardinality.

More concretely: a codim-1 foliation F (through the Frobenius theorem this is given by an integrable subbundle F of the tangent bundle TM of our 3-manifold M) is called *topologically taut* if there exists a circle S^1 which intersects transversely all leaves. A codim-1 foliation is called *geometrically taut* if there exists a Riemannian metric on M for which all leaves are minimal surfaces (ie they have mean curvature zero). One can prove that a codim-1 foliation is geometrically taut if and only if it is topologically taut. Foliations in general are very flexible structures and the taut foliations are the most rigid ones. Let us call the quotient bundle $Q := TM/F$ the *transverse bundle* to our foliation.

Let M be a Riemannian 3-manifold. Two codim-1 foliations on M are called *coarse isotopic* if up to isotopy of each one of them their oriented tangent planes differ pointwise by angles less than π. Then Gabai proves the following (Theorem 6.15 in [12]): Given any closed, orientable, atoroidal 3-manifold M with a triangulation, there exists a finite non-negative integer $n(M)$ such that any taut codim-1 foliation on M is coarse isotopic to one of the $n(M)$ taut codim-1 foliations. The condition that M should be atoroidal may be relaxed as Gabai points out. It is clear that $n(M)$ is the cardinality of the Gabai moduli space $N(M)$.

The crucial fact is that although the definition of coarse isotopy depends on the Riemannian metric, the number $n(M)$ *does not.*

Let us emphasise here that although the Gabai moduli space is finite practically for any 3-manifold, it may turn out to be *empty* [for example, S^3 has no taut codim-1 foliations].

The first idea is to try to see if one can immitate the definition of the Casson invariant using the Gabai moduli space, namely if we choose a Hegaard splitting, can we define a Casson type of invariant?

As a second step we may try to define a Floer type of theory using the Gabai moduli space. To do that we need to look for "labels" for the elements of the Gabai moduli space $N(M)$, ie *invariants for foliations.* There is one well-known invariant for codim-1 foliations, the the Godbillon-Vey invariant (see for example [14]) which is the integral over our compact 3-manifold M of the Godbillon-Vey class which for codim-1 foliations on M is a 3-dim real de Rham cohomology class defined as follows: let F be a transversely oriented codim-1 integrable subbundle of the tangent bundle TM of our closed, oriented and connected 3-manifold M. Locally F is defined by a nonsingular 1-form say ω where F consists precisely of the vector fields which vanish on ω (ie the fibre F_x where $x \in M$ equals Ker ω_x). The integrability condition of F means that $\omega \wedge d\omega = 0$. This is equivalent to $d\omega = \theta \wedge \omega$ for another 1-form θ. Then the Godbillon-Vey class is the 3-dim real de Rham cohomology class $[\theta \wedge d\theta] \in H^3(M; \mathbf{R})$. The problem however with the GV invariant is that it is only invariant under *foliation cobordisms* (see [14]) which is a *more narrow* equivalence relation than coarse isotopy, hence we may lose the finiteness of the Gabai moduli space (equivalently if we use the GV-invariant, we should restrict ourselves to only those 3-manifolds with a finite number of taut codim-1 foliations modulo foliation cobordisms; unfortunately we do not know if any such 3-manifolds exist at all).

Then the idea is to use *noncommutative geometry* techniques in order to give labels to the elements of $N(M)$. We intruduced a new invariant for foliated manifolds (see [13]) using indeed noncommutative geometry tools, in particular Connes' pairing between cyclic cohomology and K-Theory. The foliation has to be transversely oriented with a holonomy invariant transverse measure, these restrictions are quite mild. Connes' approach to foliations as described in [2] is to complete the holonomy groupoid of a foliation to a C^*-algebra and then study its corresponding K-Theory and cyclic cohomology. The invariant in [13] is constructed by defining a canonical K-class in the K-Theory of the foliation C^*-algebra and then pair it with the *transverse fundamental cyclic cocycle* of the foliation. To give a flavour of what that means we describe it in the commutative case, ie when the foliation is a fibra-

tion, in particular a principal G-bundle (where G is a nice Lie group): if we have a fibration seen as a foliation over a compact manifold (the foliated manifold is the *total space* of the fibre bundle), then this transverse fundamental cyclic cocycle is the fundamental homology class of the base manifold which is transverse to the leaves=fibres; the C^*-algebra is Morita equivalent to the commutative algebra of functions on the base manifold. By the Serre-Swan theorem the K-Theory of this commutative algebra coincides with the Atiyah topological K-Theory of the base manifold and Connes' pairing reduces to evaluating say Chern classes over the fundamental homology class of the base manifold (here we use the Chern-Weil theory to go from K-Theory to the de Rham cohomology). The key property of the canonical K-class constructed in [13] is that it takes into account the natural action of the holonomy groupoid onto the transverse bundle of the foliation. [Note: In some sense this class is similar to the canonical class in G-equivariant K-Theory, for G some Lie group acting freely on a manifold, the situation is more complicated in the foliation case since instead of a Lie group we have the holonomy groupoid of the foliation acting naturally on the transverse bundle]. We also need the result that the G-equivariant K-Theory of the total space of the principal G-bundle is isomorphic to the topological K-Theory of the quotient by the group action (since this is a G-bundle, the quotient by the G-action is the base manifold). But this invariant has not yet been properly understood: obviously if it is to be used to define invariants for 3-manifolds using the Gabai moduli space it should be invariant under coarse isotopy or under a broader equivalence relation. For the moment this point is unclear.

Another thing which seems interesting, following what we know from the commutative case, is to try to define a *Ray-Singer torsion for foliated manifolds* and then try to see if this is invariant under coarse isotopy. In order to define the Ray-Singer analytic tosrion one needs a flat connection. For the case of foliations, a flat connection always exists, it is our friend the 1-form θ appearing in the definition of the GV-class; this can indeed be seen in a natural way as a connection on the transverse bundle (for arbitrary codimension q, θ can be seen as a flat connection on the qth exterior power of the transverse bundle, this is always a line bundle). This 1-form is sometimes refered to as the (partial) *flat* Bott connection; it is flat (=closed since this is real valued ie Abelian), only when restricted to the leaf directions (which justifies the term partial; this is harmless, it can be extended to a full connection by,

for example, using a Riemannian metric).

Yet there seem to be two further possibilities here: one can also define the *tangential Laplace operator* and define its corresponding Ray-Singer analytic torsion (see [11], we shall define tangential cohomology in the next section). Yet one can use the *Cuntz-Quillen Laplacian* on cyclic cohomology (of the foliation C^*-algebra) defined in [15]. In order to define the Ray-Singer analytic torsion of the Cuntz-Quillen Laplacian one needs a convergence condition because the cyclic complex is unbounded [an idea would be to use what is called *entire cyclic cohomology* which encorporates an additional analytic convergence condition on the cyclic (co)cycles]. This Cuntz-Quillen Laplacian enabled Cuntz-Quillen to prove a *"harmonic decomposition"* theorem for cyclic cohomology, in a purely algebraic context, which is analogous to the well-known Hodge theorem for the de Rham complex on closed manifolds. (**N.B:** In the Cuntz-Quillen harmonic decomposition theorem the C-Q Laplacian does not vanish in the "harmonic part" as it happens in the Hodge theorem, it is only *nilpotent* there, but with a suitable simple normalisation we can have the normalised C-Q Laplacian vanishing in the harmonic part). That's another point which deserves further clarification.

The Heitsch-Lazarov analytic torsion in [16] is defined for foliated flat bundles and it does not seem to be of any use here since it is exactly the flat connections moduli space which we want to replace.

Note: We tend to think of foliations as generalsising *flat* vector bundles: one way to manifest the integrability of a flat connection a say is to say that its exterior covariant derivative d_a has square zero $d_a^2 = 0$, ie it is a differential. Something similar happens for foliations if one considers the "tangential" (or "leafwise") exterior derivative on the foliated manifold which is taking derivatives along the leaf directions only; this gives rise to the "tangential Laplace operator" mentioned above along with the so called *tangential cohomology* and it has corresponding *tangential Chern classes* (see [11]). In the above sense tangential cohomology can be seen somehow as a generalisation of the twisted de Rham cohomology by a flat connection. Under the light of this note the analytic torsion defined by Heitsch-Lazarov in [16] has some unsatisfactory properties for our purpose since it is a torsion for a foliated flat bundle (namely a flat bundle whose base sapce is in addition,

foliated, and so the total space carries essentially 3 structures : the fibration, the foliation where the leaves are covering spaces of the base space–flatness– and another foliation which under the bundle projection projects leaves to leaves.

The 3rd point which is the most ambitious is to try to define a sort of Floer homology using the Gabai moduli space. In order to do that one needs to develop a Morse theory for foliated manifolds. One has at first to find a Morse function whose critical points will be the taut codim-1 foliations. Immitating perhaps naively the Floer homology case we have two natural candidates for a Morse function: tangential Chern-Simons forms and Chern-Simons forms for cyclic cohomology as developped by Quillen not very long ago in [15] (that's a noncommutative geometry tool). The hope is that by using the Gabai moduli space one might have a chance to avoid the problems with reducible connections (ie the "bubbling phenomenon", see [8]) when trying to define Floer homology groups for 3-manifolds which are not homology 3-spheres. There are some more versions of Floer homology available but they need some extra structure: a $spin^c$ structure for the Seiberg-Witten version (and use of the monopole equation instead of the instanton equation), or a symplectic structure (as in the original Floer attempt) or a complex structure (as in the Oszvath-Szabo approach where one uses complex holomorphic curves instead of instantons).

One of the problems one faces in the above is that there seem to be at least three cohomology theories which can describe foliated manifolds (more precisely the space of leaves of a foliated manifold): the tangential cohomology, the cyclic cohomology of the corresponding C^*-algebra of the foliation and the so-called *Haefliger cohomology* (which has been used in the construction of the Heitsch-Lazarov analytic torsion). In ordinary Morse theory, given a compact smooth manifold, one considers a real valued function (called the Morse function) defined on the manifold and under favourable cases one can reconstruct the homology of the manifold by using the flows of the critical points of the Morse function. In a would-be Morse theory for foliated manifolds one would like to reconstruct the homology of the space of leaves using a suitable Morse function, but it is currently unclear which homology of the 3 above is more suitable! Moreover the critical points should correspond to taut foliations in order to use the Gabai moduli space. We think

the above challenge is fascinating. Some progress towards a Morse theory for foliated manifolds has been already made by Connes-Fack. Also N. Nekrasov et al. work on *noncommutative instantons* may give hints on how to preceed building a noncommutative Floer Homology using the Gabai moduli space of isotopy classes of taut codim-1 foliations.

In order to make some progress towards this direction we need at least 2 tools: a *Morse theory* (finite dimesnional at a first stage) for *foliated manifolds* and analytic torsion for foliated manifolds. We shall say something more concrete about the first in the next section.

3 Morse Theory for Measured Foliations

Let (M, F) be a smooth foliation on a closed n-manifold M (and F is an integrable subbundle of the tangent bundle TM of M where $dimF = p$, $codimF = q$ with $p + q = n$), equipped with a holonomy invariant transverse measure Λ (we need that in order to be able to perform the analogue of "integration along the fibres" which we do for vector or principal G-bundles using the Haar measure which is invariant under the group action). We consider the *tangential cohomology* coming from the differential graded complex $(d_F, \Omega^*(M, F))$, where d_F denotes the *tangential* exterior derivative (namely taking derivatives only along the tangential (leaf) directions) and $\Omega^*(M, F)$ denotes forms on M with values on the bundle F. Due to the integrability of F, the tangential exterior derivative is also a differential, namely $d_F^2 = 0$, hence we can take the cohomology of the above complex. We pick a Riemannian metric g on M (which, when restricted to every leaf gives a Riemmanian metric on every leaf), we consider the adjoint operator d_F^* and we form the *tangential Laplacian* $\Delta_F := d_F^* d_F + d_F d_F^*$. We know that a *tangential Hodge theorem* holds, hence Ker $[\Delta_F^k]$ (tangential Laplacian acting on tangential k-forms) captures the tangential cohomology groups (see [2]). We denote by β_k the k-th tangential Betti number ($0 \leq k \leq p$), where clearly $\beta_k = dim_\Lambda[Ker(\Delta_F^k)]$. We must make an important remark here: this is the Murray-von Neumann dimension defined by Connes using the invariant transverse measure, it is finite; the tangential cohomology groups may be infinite dimensional as linear spaces (see [11]).

Now let ϕ be a smooth real valued function with domain the manifold M. A point $x \in M$ is called a *tangential singularity* of ϕ if $d_F\phi(x) = 0$. A tangential singularity is called a *tangential Morse singularity* if it is nondegenerate, namely the tangential Hessian $d_F^2\phi(x)$ is nonsingular. The *index* of a t-Morse singularity (in the sequel, "t" stands for tangential), is defined as the number of "-" signs in the signature of the Hessian (quadratic form) $d_F^2\phi(x)$. We denote by $S_F(\phi)$, $S_{1,F}(\phi)$ and $S_{1,F}^i(\phi)$ the set of all tangential, t-Morse and t-Morse of index i singularities of ϕ respectively. We have that $S_F(\phi)$ is a closed q-submanifold of M which is *transverse* to F. A *tangential Morse function* is one with only tangential Morse singularities, or equivalently it is a function which is a Morse function when restricted to every leaf. This definition covers our application for 3-manifolds and the Gabai moduli space of taut codim-1 foliations. Yet in general it is a rather restrictive definition since there are many interesting foliations with no closed transversals. For that, we give the definition of an *almost tangential Morse function* ϕ w.r.t. the holonomy invariant transverse measure Λ to be one for which the set $\{x \in M$ s.t. $\phi|L_x$ has only t-Morse singularities $\}$ is Λ-negligible (where L_x denotes the unique leaf through the point x).

It is then not very hard to see that $c_k := \Lambda(S_{1,F}^k(\phi)) < \infty$. The main results of Connes and Fack in [2] are the following two theorems:
Theorem 1; If $q \leq p$ and for any *good tangential almost Morse function*, one has the *tangential weak Morse inequalities*:

$$\beta_k \leq c_k$$

Theorem 2; Under the same assumptions as above, one has the *tangential Morse inequalities*:

$$\beta_0 \leq c_0,$$

$$\beta_1 - \beta_0 \leq c_1 - c_0,$$

$$...,$$

$$\beta_k - \beta_{k-1} + ... + (-1)^k\beta_0 \leq c_k - c_{k-1} + ... + (-1)^k c_0$$

These results are proved based on some hard analytic results of Igusa and his *Parametrised Morse Theory* (see [18]) and by using a Witten type (see

[7]) of *perturbed tangential* Laplacian $\Delta_{F,\tau} := d^*_{F,\tau} d_{F,\tau} + d_{F,\tau} d^*_{F,\tau}$ by a good tangential almost Morse function ϕ: $d_{F,\tau} := e^{-\tau\phi} d_F e^{\tau\phi}$, where τ is a positive real parameter.

We must explain the term good tangential almost Morse function: following the pioneering work of K. Igusa (see [18]) and his *Parametrised Morse Theory*, Connes and Fack managed to prove the above results not only for tangential Morse functions but for a larger class of real valued smooth functions with domain a foliated manifold: a good tangential almost Morse function ϕ is a *generalised* (namely it can contain Morse singularities and *birth-death* singularities), almost tangential Morse function which is *generically unfolded*. The last requirement means that there exist *normal forms* describing the function in a neighbourhood of a birth-death singularity. A birth-death singularity is a degenerate tangential singularity (i.e. tangential Hessian vanishes) for which the restriction of the map $x \mapsto (d_F(\phi)(x), det[d^2_F(\phi)(x)])$ has rank p at x.

Let us make the following remarks here: Connes and Fack, before proving the tangential Morse inequalities, they prove that *every* measured foliation with $q \leq p$ has at least one good tangential almost Morse function; their proof is based on an astounding theorem due to K. Igusa: it was a well-known fact that a generic smooth real valued function on a closed manifold has only nondegenerate critical points; however a generic 1-parameter family of real valued smooth functions has in addition birth-death points where critical points are created or canceled in pairs. A multi-parameter family has a zoo of complicated singularities; K. Igusa proved that more complicated singularities can be avoided: for any foliation on a closed manifold it is always possible to find a smooth real valued function such that singularities associated with the critical points of its restriction to every leaf are at most of degree 3! Clearly we think of a foliation as a more complicated parametrised family of manifolds than a fibre bundle: the family of manifolds (leaves-they correspond to the tangential directions) is parametrised by the space of leaves (corresponds to the transverse directions); in a fibre bundle we have a family of manifolds (fibre) parametrised by the base manifold.

It is not true that any measured foliation with $q \leq p$ has a tangential Morse function, namely the foliations with tangential Morse functions are rather special (they must have a closed transversal); taut foliations never-

theless, which is what we are mostly interested in, do have, by definition, a complete closed transversal).

If we denote by $A(M,F)$, $J(M,F)$ and $R(M,F)$ the sets of tangential almost Morse functions, tangential generalised Morse functions and tangential generalised Morse functions which are generically unfolded respectively, then the good tangential almost Morse functions are those in the intersection of $A(M,F)$ and $R(M,F)$ (clearly the 3rd set is a subset of the second). The hard piece due to K. Igusa is to prove that for a closed M and an F with $codimF \leq dimF$, the set $J(M,F)$ is nonempty.

For ϕ a good tangential almost Morse function, we have that the critical manifold $S_F(\phi)$ is a q-dim submanifold of M transverse to F, the set of tangential Morse singularities of index i $S^i_{1,F}(\phi)$ is also a q-submanifold of M transverse to F and open inside the critical manifold (but not closed in M in general) and the set of tangential birth-death singularities $S^i_{2,F}(\phi)$ of index i of ϕ is a closed $(q-1)$-submanifold of the critical manifold and it is both in the closure of $S^i_{1,F}(\phi)$ and of $S^{i+1}_{1,F}(\phi)$.

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